Report on the article: 'Impact of a block structure on the Lotka-Volterra model'

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Summary

The article is presenting a Lotka-Volterra model with a large number of interacting species structured by blocks. Some theoretical heuristics are given and many numerical simulations are done to study how interactions parameters affect the survival of the community. The model corresponds to a system of a large number of coupled EDOs, describing the density of the species. The interactions are modelled through a random matrix of independent centered Gaussian entries structured by blocks. Each block is characterized by a parameter s_{ij} that defines the variance of the r.v.

The authors study properties of existence of equilibrium, global stability and feasibility (positivity of all components of the equilibrium) depending on the values of the variances. They do not always give exact results, but heuristics are well justified and illustrated with simulations.

I found the paper interested, well written and really easy to read. The subject is very relevant, and several advances have recently been made in the study of this type of large community modeled via a random matrix. The extension to a block structure seems natural, and the first results described in this paper show the interest of using this kind of method to understand the dynamics of biological communities, which are often very large and complex. In particular, the authors are able to provide some quantified behaviours. Finally, the paper opened many perspectives that could be interested for future works.

The only point that requires a little consideration before publication is a clarification regarding the use of QVE theory. Indeed, the authors do not specify the links between this theory and the eigenvalues of matrices, which makes it difficult to read and understand the proof of Theorem 3 or Appendix B, for example. The authors should add an appendix on the subject or a paragraph with precise references.

Finally, besides this and the comments and (very few, thanks!) typos that I give below, I think that this paper can be accepted for publication.

Comments

1. p.3: clarifies $\mapsto$ simplifies ?

2. p.3: The two equivalences at the top of p.2 do not seem obvious for me. The authors should add more details.

3. p.3: "the intra-communities interactions are small enough": enough with respect to what ?

4. p.3: "It is no longer possible for both communities to maintain...": Is it true almost surely ? Or is it true for the special realization the authors did ? They should specify a bit more.

5. p.4: "outline of the article": the authors should specify that some of their results are in fact heuristics and not exact proofs (although well justified).

6. p.5 (6): add "$\forall k \in [n]$"

7. p.5 (7): This inequality is not clear: what are the values of the other coordinates of x? I think that this should be true only for species such that $x_k^* = 0$. If it is the case, the authors should add the mention.

8. p.6 Proposition 1: I do not see the interest of this proposition here. Do they need it for the proof of Theorem 2? In this case, the proposition should be given closer to this Theorem or even suppress.

9. p.6 Theorem 3: $I - B \in \mathcal{D}$ should be $B - I \in \mathcal{D}$.

10. p.6 l.-2: "$(-I + B) \in \mathcal{D}$": the authors do not know this fact at that time since they want to prove it. They should suppress $\in \mathcal{D}$.

11. p.7 QVE: What does z belong to?

12. p.8: Hereafter, we describe "heursitics of the" statistical properties of x^*...

13. p.8: before Heuristics 1:, the authors should give a sentence to explain what they will present in their heuristics, that they will illustrate some of the heuristics with simulations and that they will then give details on how to find the heuristics.

14. p.9: the fact that $\hat{p}_i \underset{n\to\infty}{\to} p_i^*$ and $\hat{\sigma}_i \underset{n\to\infty}{\to} \sigma_i^*$ is not part of their heuristics, if I understand well. It's more like a hypothesis they're making or a conjecture, but there's no intuition given to back up this precise result. This should be specified.

15. p.9: In Figure 3, the authors should refer to Appendix B

16. p.10: construction of the heuristics: at the beginning of the reasoning, I was a bit confused between what's assumed, what's proven, what's an approximation and what's an exact equality. All this should be taken up again to be clearer, keeping the = sign for "true" equalities.

17. p.10; The authors could use more the notation Δ_i^* that they introduced in order to simplify the reading.

18. p.10: "Heuristics (12)-(13)": there are several typos in the two following equations that follow. $\geq \mapsto >$; $\mathcal{S} \mapsto \mathcal{I}$; $\leq \mapsto <$.

19. Section 3.3: Can the authors comment a bit more the results of this section from a biological point of view ?

20. p.12 l.1: by $\delta_i^* = \delta_i(p_i^*, \sigma_i^*)$ "and $\Delta_i^* = \Delta_i(p_i^*, \sigma_i^*)$".

21. p.12 The authors could refer to Equation (17) when writing about $x_k^* = 1 + \Delta_i^* Z_k$.

22. Figure 5: Can the authors add mean values with dashed lines as in Figure 4 ?

23. p.14 l.2: Can you specify that the equilibrium in this case is thus the feasible equilibrium?

24. Theorem 4: In my sense, it is not really a theorem but a heuristic or a conjecture. The authors should specify this fact.

25. p.14: "I"n the critical regime

26. p.14: Why they are introducing κ in this way? Using a definition similar to the γ, i.e. $\kappa = \sqrt{n}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ should be easier to interpret. In particular, the variations would be the same as s and the explanations in 4.2 should be easier to follow.

27. p.15, section 4.2: recall that $\beta_1 = 1 - \beta_2$.

28. Figure 7: what do the values of the color axis correspond?

29. Section 4.3: could they detail what happens if $(\kappa_{11}, \kappa_{22})$ are greater than $(\kappa_{12}, \kappa_{21})$?

30. p.20: Discussion "and perspectives" ?

31. Appendix A: what is $\nu(a, b)$? This is not clear

32. Appendix B: the authors could add subsections.

33. p.27: The beginning of this page is not clear: why can they simplify assuming that $m_k(z) = \mu(z)$ or $\nu(z)$? They should give more details. Moreover, they should give of precise reference when they are writing about RMT.

34. p.27 using "~~a~~" the

35. Section C.1: they authors should recall that (x_k^*) are assuming to be independent from (B_{kl}), which is a strong assumption, even though it is probably true asymptotically.

36. p.28: I do not see the interest of the equation below the figure.

37. p.30: the authors should give a reference or more details to get equation (27).