REVISION OF THE PAPER "IMPACT OF A BLOCK STRUCTURE ON THE LOTKA-VOLTERRA MODEL" BY CLENET, MASSOL AND NAJIM.

Dear Editor,

We are excited to submit a revised version of our manuscript, "Impact of a block structure on the Lotka-Volterra model" by Maxime Clenet, François Massol and Jamal Najim. We appreciate the opportunity to improve our manuscript by incorporating the feedback provided by the Reviewers. These revisions have helped us improve the clarity of the manuscript.

Please find the revised manuscript attached, and a detailed point-by-point response to all comments below.

Sincerely,

Maxime Clenet, on behalf of the Authors.

Note: Reviewer comments are in blue. Please be advised that the code has been updated and the number of the figures has been updated.

Reviewer #1

General comments.

(1) *The Introduction needs improvement to make the paper more accessible and appealing to readers unfamiliar with the topic. The introduction should provide an overview of the aims and the main results and contributions of the paper, and should not immediately launch into technical details. The introduction (see the subsections 'Model and assumptions' and 'Properties of the dynamical system') mentions some technical aspects (such as existence of a unique equilibrium and its asymptotic stability), some of which could be delayed to a later stage in the paper. It would also be good to provide an explicit biological motivation for the generalized setting investigated in this paper. Perhaps some of the summary statements made in the Discussion section should be moved to the introduction.*

We appreciate the reviewer's suggestions regarding the introduction and have implemented the following changes as a result:

- We've added two paragraphs to clearly separate the introduction from the broader context: a "motivations" paragraph and a "known results" paragraph.
- We have modified the first paragraph of the "motivations" section of the Introduction to include more ecological context:

"Understanding large ecosystems and the underlying mechanisms that support high species diversity is a major challenge in theoretical ecology. Since the 1950's, understanding how species can stably coexist has been the focus of both theoretical [16, 17, 15, 22, 12] and empirical studies [27, 25, 14]. Motivated by the seminal work of May [18], the introduction of random matrices has been a key mathematical step in modeling high-dimensional ecosystems [3, 28, 2]. These tools have expanded our ability to understand the nature of interactions and how food webs can recover after small perturbations (stability) [3, 29]. In the course of the debate on the theory of species stable coexistence, a number of questions have emerged, including the following: What are the conditions which enable many species to coexist, especially regarding the structure of their interaction matrix?"

- We have added the paragraph "known results" to be consistent with the discussion and to better contextualize our results with the existing literature:

  " Building upon the insights gained from the model of May [18], an understanding of the Lotka-Volterra model provides a foundation for in-depth analysis of the impact of interactions on community dynamics. Scientists from diverse disciplinary backgrounds, including mathematics, physics, and ecology, sought to investigate the intricacies of this complex ordinary differential equations system. The study of the stability of the Lotka-Volterra model has constituted a central focus of research, as evidenced by the works of Stone [28] and Gibbs *et al.* [11], which have been complemented by Clenet *et al.* [9] where they investigated the properties of a stable equilibrium.
  In fact, beyond the stability of the equilibria, the properties of these equilibria have been the subject of central interest [26, 19, 20], e.g. deriving the number of surviving species. Moreover, the existence of a feasible equilibrium and its stability have been demonstrated by Bizeul *et al.* [5] where they establish that a threshold of interaction strength exists beyond which equilibrium of the system is almost certainly feasible. The methodology was further refined in the case of sparse interactions [1] and a correlation profile [8]. In order to gain a more comprehensive understanding of the LV model, Akjouj *et al.* [2] conducted a comprehensive mathematical review of the subject.
  Lotka-Volterra model provides an interesting diversity of dynamical behaviors, with partial mathematical knowledge. This is supplemented by methods from physics to improve the understanding on these various dynamical behaviors (properties of the equilibrium, out-of equilibrium dynamics, model sophistication). Bunin [7, 6] used the cavity methods to derive the properties of the surviving species and the multiple attractors phase. Barbier *et al.* [4] exhibits generic behaviors in complex communities.
  Generating functional techniques for deriving similar mean-field equations to study the equilibrium phase in the LV system was used by Galla [10] and extended by Poley *et al.* [21] to study the LV model in the case of a cascade interaction matrix. "

(2) *In May's work (see the references [May72] and [AT12] of the manuscript), the model is written*

$$\frac{dx_k}{dt} = x_k\left(r_k - \theta_k x_k + \sum_{l\neq k} B_{kl}x_l\right),\ k \in \{1,\cdots,n\}$$

where x_k is the density of species k, r_k represents its intrinsic growth rate, θ_k is an intraspecific feedback coefficient (most often denoted $\theta_k = r_k/K_k$, where K_k is the carrying capacity) and B_{kl} is the per capita effect of species $l \neq k$ on species k. The off diagonal coefficients B_{kl} of the random matrix interaction are drawn from a normal distribution. In this paper, see model (1), it is assumed that $\theta_k = 1$, i.e. the carrying capacity is equal to the intrinsic growth rate (what is the biological significance of such an hypothesis ?) and in addition the intraspecific coefficient B_{kk} is not equal to 0 but random. These important differences between the models that are used in the literature and the model (1) studied by the authors should be discussed.

- We have fixed the intraspecific parameter $\theta_k = 1$ because we have omitted the computations to obtain the dimensionless LV model. In particular, in the case

$\theta_k \neq 1$, we make a modification of the variables of the system

$$\frac{dx_k}{dt} = x_k \left(r_k - \theta_k x_k + \sum_{l \neq k} B_{kl} x_l \right), \; k \in \{1, \cdots, n\}$$

to avoid the parameter θ_k by setting $\tilde{x}_k := \theta_k x_k$, $\tilde{B}_{k\ell} := B_{k\ell}/\theta_k$ and we get again model (1) in the paper.
We have added the following sentence in the paper after the introduction of Model 1 for further clarification: "The intraspecific parameter has been set to 1 in accordance with the computations that were conducted in order to obtain a dimensionless LV model (for further details, please refer to Remark 2.1 in Akjouj *et al.* [2])."

- The normalization of the diagonal coefficient of the matrix B_{kk} by $\sqrt{n}$ implies that it has a microscopic effect on the intraspecific coefficient $\theta = 1$. This approach is supported by numerous publications, including Allesina & Tang [3] i.e. the selection of zero off-diagonal coefficients has no effect on the outcome. Furthermore, it enables the application of rigorous and well-referenced random matrix theory.

(3) *The authors focus on the case with equal growth rates. See the model (3). What is the biological significance of this restriction ?*

We agree with the reviewer that the choice of $r = 1$ has a non-conventional biological context. It is important to note that the model under consideration should be viewed as a toy model. In this paper, our primary interest is the impact of the interaction matrix on the Lotka-Volterra model. Selecting equal growth rates releases the total number of parameters, greatly simplifies the computational processes, and reduces the complexity of the model. Other researchers have explored the potential for coexistence given different growth rates [23, 24].

However, to make the reader aware of cases where the choice of growth rates may be different from 1, the footnote has been expanded according to the Specific Remark (4):

" The simplifying assumption $r_k = 1$ allows tractable computations and could be extended to $r_k = c$ with $c > 0$. However, if the growth rate is different for each species, the mathematical development and result may be strongly affected and will be discussed in a series of remarks, see Remarks 2, 4 and 5."

Specific remarks.

(1) *Page 2: The following sentence is unintelligible "where [...]" I suggest rewording as follows [...].*

We agreed with the reviewer that the sentence was unintelligible. We have revised the sentence accordingly with the proposition.

(2) *Page 2. What do you mean by "The Gaussianity assumption clarifies the explanations, but can be relaxed under certain circumstances (see the corresponding sections for details)." Please specify the sections where this is done.*

We have updated the sentence in question with a specific reference to the remarks where this is discussed in more detail. The sentence is now:

" The Gaussianity assumption simplifies the explanations, but can be relaxed under certain circumstances (see the series of remarks, specifically Remarks 2, 4 and 5, which can be found in their respective sections for further details). "

(3) *Page 2. The normalization parameter $1/\sqrt{n}$ in the matrix B in (2) deserves further explanation. The sentences "This enables the interaction matrix B to have a macroscopic effect on system (1)..." and "From an ecological perspective, an increase in the number of species may not necessarily lead to a corresponding increase in the overall strength of*

interactions between one species and all others." on page 2, lines 6 to 10, deserve more detail to make them comprehensible to a reader who is not a specialist in the subject

The first writing was a bit harsh indeed. In the new version, we developed the explanation and wrote:

Notice a normalization parameter $1/\sqrt{n}$ in the matrix B. This enables the interaction matrix B to have a *macroscopic effect* on system (1). By macroscopic effect, we mean that even if the number of species n grows to infinity, the effect of matrix B in Eq. (1) remains noticeable (it does not vanish, nor does it explode). This can be illustrated by the following asymptotic properties of B (hereafter $\|\cdot\|$ stands for the spectral norm):

$$\|B\| \sim O(1)\ ; \qquad \mathbb{E}\left(\sum_{\ell\in[n]} B_{k\ell}x_\ell(t)\right) \sim O(1)\ ; \qquad \mathrm{Var}\left(\sum_{\ell\in[n]} B_{k\ell}x_\ell(t)\right) \sim O(1)$$

as $n \to \infty$. From an ecological perspective, this normalization has the following consequence: an increase in the number of species does not yield a corresponding increase in the overall strength of interactions between one species and all others, which order of magnitude remains similar.

(4) *Page 2. The assumption $r_k = 1$ is very restrictive from an application point of view. Please clarify the footnote "The simplifying assumption $r_k = 1$ allows tractable computations and could be extended to $r_k = c$ with $c > 0$. However, if the growth rate is different for each species, the mathematical development and result may be strongly affected and will be discussed in each section." Please specify the sections where this is done.*

We have updated the sentence in question with a specific reference to the remarks where this is discussed in more detail. Please find the revised sentence:

"However, if the growth rate is different for each species, the mathematical development and result may be strongly affected and will be discussed in a series of remarks, see Remarks 2, 4 and 5."

Reviewer #2

(1) *The title reflects the content of the article. It could be made more precise such as: Impact of a block structure on equilibria of Lotka-Volterra Model.*

We would like to thank Reviewer #2 for taking the time to read the paper carefully and provide us with constructive feedback on various specific points in the article. In particular, we appreciate the feedback provided by Reviewer #2 regarding the title. We concur that it is important to highlight that our study primarily concerns equilibrium properties. However, we also provide insights into the dynamics of the system in Section 3 on the existence and uniqueness of the equilibrium in the LV system. Furthermore, we would like to retain the simplicity of the title, as any changes would likely complicate the administrative procedure.

Reviewer #3

General comments.

(1) *The only point that requires a little consideration before publication is a clarification regarding the use of QVE theory. Indeed, the authors do not specify the links between this theory and the eigenvalues of matrices, which makes it difficult to read and understand the proof of Theorem 3 or Appendix B, for example. The authors should add an appendix on the subject or a paragraph with precise references.*

We have added a paragraph in the beginning of Section 2.2 which hopefully explains the overall strategy of the proof of Theorem 2 (labeled in the previous version as Theorem

3). We have also slightly rewritten the proof of this Theorem, which should be clearer now.

Comments.

(1) *p.3: clarifies $\rightarrow$ simplifies ?*

This has been fixed.

(2) *p.3: The two equivalences at the top of p.2 do not seem obvious for me. The authors should add more details.*

We have added more details.

(3) *p.3: "the intra-communities interactions are small enough": enough with respect to what ?*

We do agree with the reviewer that the matter in question was not sufficiently clear. We suggest the following revised wording:

"and the intra-communities interactions were selected to be relatively small (Fig. 1a) in order to observe that both communities' dynamics converge to a feasible equilibrium, in which all species survive (Fig. 1b).".

(4) *p.3: "It is no longer possible for both communities to maintain...": Is it true almost surely ? Or is it true for the special realization the authors did ? They should specify a bit more.*

We agree with the reviewer that the original text was not entirely clear. To provide a more detailed and unambiguous explanation, we have rewritten the second scenario paragraph as follows:

" In the second scenario, we increase the interactions between the communities (Fig. 2a), i.e. the standard deviation matrix is defined by

$$\boldsymbol{s} = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}.$$

In the case of a given inter-communities interaction realization, such as the one shown in Fig. 2, it is no longer possible for both communities to maintain the feasibility of all species. Some species are likely to disappear (Fig. 2b). "

(5) *p.4: "outline of the article": the authors should specify that some of their results are in fact heuristics and not exact proofs (although well justified).*

We concur with the reviewer's assessment and offer the following clarifications. We have revised the sentence in question as follows:

"Section 3 is devoted to the study of the properties of the species that survive in each of the communities described by two Heuristics. Heuristics 1 specify the properties of the surviving species and Heuristics 2 define the distribution of the surviving species."

(6) *p.5 (6): add "$\forall k \in [n]$"*

This has been fixed.

(7) *p.5 (7): This inequality is not clear: what are the values of the other coordinates of x? I think that this should be true only for species such that $x_k^* = 0$. If it is the case, the authors should add the mention.*

We concur with the reviewer's assessment that the sentence in question was incorrect. The sentence:

" In model (3), the non-invadability condition for a given species $k \in [n]$ is equivalent to "

has been changed to

" In model (3), the non-invadability condition for a given species whose values at equilibrium is $x_k^* = 0$ is equivalent to "

Note that when $x_k^* > 0$, by definition of equilibrium, we have $\frac{dx_k}{dt} = 0$.

(8) *p.6 Proposition 1: I do not see the interest of this proposition here. Do they need it for the proof of Theorem 2? In this case, the proposition should be given closer to this Theorem or even suppress.*

We agree with the reviewer that the interest of Proposition 1 is not justified. Indeed, it is not necessary to define the notion of P-matrix for the proof of Theorem 2. As a result, Proposition 1 has been removed.

(9) *p.6 Theorem 3: $I - B \in D$ should be $B - I \in D$.*

This has been fixed by replacing $I - B \in D$ by $-I + B \in D$

(10) *p.6 l.-2: "$(-I + B) \in D$": the authors do not know this fact at that time since they want to prove it. They should suppress $\in D$.*

This has been fixed.

(11) *p.7 QVE: What does z belong to?*

We agree with the reviewer that the original text was not sufficiently clear. We have amended the sentence introducing the functions m(z) to read: "Given $\boldsymbol{m}(z) = (m_1(z), \cdots, m_n(z))$, $z \in \mathbb{C}^+$".

(12) *p.8: Hereafter, we describe "heuristics of the" statistical properties of x^*...*

This has been fixed.

(13) *p.8: before Heuristics 1:, the authors should give a sentence to explain what they will present in their heuristics, that they will illustrate some of the heuristics with simulations and that they will then give details on how to find the heuristics.*

As outlined by the reviewer, we have added a brief introductory paragraph before presenting Heuristic 1:

" In Heuristic 1, we derive the properties of surviving species, specifically the proportion of surviving species and the mean square of surviving species. The presentation of these properties is illustrated in Fig. 3 through the use of numerical simulations to support this heuristic. In addition, the technical details of how to obtain the heuristic are presented in the subsequent paragraphs. "

(14) *p.9: the fact that $\hat{p}_i \xrightarrow[n\to\infty]{} p_i^*$ and $\hat{\sigma}_i \xrightarrow[n\to\infty]{} \sigma_i^*$ is not part of their heuristics, if I understand well. It's more like a hypothesis they're making or a conjecture, but there's no intuition given to back up this precise result. This should be specified.*

We agree with the reviewer. Both convergences are not a heuristics but rather a conjecture whose justification follows from the fact that p_i^* and σ_i^* are computed starting from the empirical definition of $\hat{p}_i$ and $\hat{\sigma}_i$. We clarified this in the main document (see Remark 4 after Heuristics 1).

(15) *p.9: In Figure 3, the authors should refer to Appendix B*

We added the following sentence at the end of the legend of Figure 3: "For any numerical details, please refer to Appendix B".

(16) *p.10: construction of the heuristics: at the beginning of the reasoning, I was a bit confused between what's assumed, what's proven, what's an approximation and what's an exact equality. All this should be taken up again to be clearer, keeping the = sign for "true" equalities.*

We agree with the reviewer. We tempted to clarify the writing of Section 3.2 by adding details. On two occasions, we precised "this assumption is part of the heuristics" and "this argument is part of the heuristics" together with other details. We hope that this new writing is clearer.

(17) *p.10; The authors could use more the notation Δ_i^* that they introduced in order to simplify the reading.*

In the section, construction of the heuristics, two equations was modified:

•
$$\mathrm{Var}(\check{Z}_k) = p_1^*(\sigma_1^*)^2\beta_1 s_{i1}^2 + p_2^*(\sigma_2^*)^2\beta_2 s_{i2}^2\,,$$

where p_1^*, p_2^*, σ_1^*, σ_2^* are resp. the limits of $\hat{p}_1$, $\hat{p}_2$, $\hat{\sigma}_1$, $\hat{\sigma}_2$.
to

$$\mathrm{Var}(\check{Z}_k) = (\Delta_i^*)^2\,,$$

where Δ_i^* corresponds to the average variance of the interactions on community i which depends on four parameters p_1^*, p_2^*, σ_1^*, σ_2^*. The latter are resp. the limits of $\hat{p}_1$, $\hat{p}_2$, $\hat{\sigma}_1$, $\hat{\sigma}_2$.

-
$$\forall k \in \mathcal{I}_i, \quad Z_k = \frac{\check{Z}_k}{\sqrt{\mathrm{var}(\check{Z}_k)}} = \frac{\check{Z}_k}{\sqrt{p_1^*(\sigma_1^*)^2\beta_1 s_{i1}^2 + p_2^*(\sigma_2^*)^2\beta_2 s_{i2}^2}}.$$

to

$$\forall k \in \mathcal{I}_i, \quad Z_k = \frac{\check{Z}_k}{\sqrt{\mathrm{Var}(\check{Z}_k)}} = \frac{\check{Z}_k}{\Delta_i^*}.$$

(18) *p.10: "Heuristics (12)-(13)": there are several typos in the two following equations that follow.* $\geq \mapsto >; S \mapsto I; \leq \mapsto >$.

We thank the reviewer for pointing out this error, which has now been corrected. Please also note that a similar error was discovered during proofreading in Section 3.3 on the proportion of surviving species in the general community. This error has also been corrected.

(19) *Section 3.3: Can the authors comment a bit more the results of this section from a biological point of view ?*

To provide further context for the following sentence, "We observe the linear effect of community size $\boldsymbol{\beta}$ on general properties," we have added a paragraph. The following paragraph has been added to end Section 3.3:

"This linear relationship illustrates that the impact of an ecological community on the benefits of the entire ecosystem is directly proportional to its size. In other words, a larger ecological community will have a greater influence on the ecosystem than a smaller ecological community."

(20) *p.12 l.1: by* $\delta_i^* = \delta_i(p_i^*, \sigma_i^*)$ *"and* $\Delta_i^* = \Delta_i(p_i^*, \sigma_i^*)$*"*

This has been fixed.

(21) *p.12 The authors could refer to Equation (17) when writing about* $x_k^* = 1 + \Delta_i^* Z_k$.

This has been fixed by replacing

"The heuristics simply follows from the fact that if x_k^* is a surviving species and $k \in \mathcal{S}_i$ then..."

by

"The heuristics are derived from the fact that, from equation (17), if x_k^* is a surviving species and $k \in \mathcal{S}_i$ then..."

(22) *Figure 5: Can the authors add mean values with dashed lines as in Figure 4 ?*

The dashed lines have been incorporated into Figure 5. In addition, the legend has been revised, and the code and figure on GitHub have been updated as a consequence.

(23) *p.14 l.2: Can you specify that the equilibrium in this case is thus the feasible equilibrium?*

This has been fixed by replacing

" Note that for sufficiently large n, the problem satisfies the sufficient condition of Theorem 3 to have a unique globally stable equilibrium. "

by

" Note that for sufficiently large n, the problem satisfies the sufficient condition of Theorem 3 to have a unique globally stable equilibrium, which in this case is a feasible equilibrium. "

(24) *Theorem 4: In my sense, it is not really a theorem but a heuristic or a conjecture. The authors should specify this fact.*

Notice that former Theorem 4 is now Theorem 3 (we have removed former Proposition 1 in this new version). This is in fact a theorem and it could be proven and written in full mathematical details if needed (at the cost of 10+ pages). We rather prefer to provide a sketch of proof in Appendix D, which illustrates the main ideas.

This is carefully explained in this version of the article.

(25) *p.14: "I"n the critical regime*

This has been fixed.

(26) *p.14: Why they are introducing* κ *in this way? Using a definition similar to the* γ*, i.e.* $\kappa = \sqrt{n}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ *should be easier to interpret. In particular, the variations would be the same as s and the explanations in 4.2 should be easier to follow.*

The decision to introduce the $\boldsymbol{\kappa}$ matrix in this way ensures that our approach remains consistent with the existing bibliography on the study of feasibility in the LV model. This was first developed by Bizeul *et al.* [5] and subsequently extended [1, 8] to cover more complex interaction matrices. It should be noted that modifying this notation would entail a significant amount of reworking.

(27) *p.15, section 4.2: recall that* $\beta_1 = 1 - \beta_2$.

This has been fixed.

(28) *Figure 7: what do the values of the color axis correspond?*

The values on the colored axis correspond to the y-axis values, to allow better visualization of the variation of the inter-community interactions. To provide further clarification, the final sentence of the legend has been replaced with the following:

" The colored area, where the gradient of color represents the strength of inter-community interactions (same values of the y-axis), illustrates the threshold between the co-feasible and non-co-feasible domains in the system. "

(29) *Section 4.3: could they detail what happens if* $(\kappa_{11}, \kappa_{22})$ *are greater than* $(\kappa_{12}, \kappa_{21})$*?*

We have not gone into the details of these calculations in order to keep our results simple, but we agree with the reviewer that this case is also interesting.

In order to provide further clarification regarding the implications of $(\kappa_{11}, \kappa_{22})$ exceeding $(\kappa_{12}, \kappa_{21})$, we have included a new remark at the conclusion of the section, accompanied by a graphical illustration (Figure 9) to reinforce our argument:

" If $(\kappa_{11}, \kappa_{22})$ are greater than $(\kappa_{12}, \kappa_{21})$, there may be different situations depending on the value of $(\kappa_{11}, \kappa_{22})$, i.e. if $(\kappa_{11}, \kappa_{22}) > \sqrt{2}$ are large, we can have co-feasibility (see Fig.9a), whereas if $(\kappa_{11}, \kappa_{22})$ are small, we may not have co-feasibility (see Fig. 9b). From a biological perspective, we believe that the case $(\kappa_{11}, \kappa_{22})$ smaller than $(\kappa_{12}, \kappa_{21})$ is more significant because the interactions within each community are stronger than those between communities. "

Please be advised that the code has been updated and the number of the figures has been updated.

(30) *p.20: Discussion "and perspectives" ?*

This has been fixed. We agree that adding "and perspectives" in the context of an applied mathematics paper is consistent with the standards and will inform the reader of the more specific content of this final section.

(31) *Appendix A: what is* $\nu(a, b)$*? This is not clear.*

In Appendix A, ν is a measure in the mathematical sense while in Appendix B, ν is a Stieltjes transform, which is rather confusing. In order to avoid this, we use notation ν for measures (Appendix A), and notations m and $\check{m}$ for Stieltjes transforms in Appendix B.

ν is a measure in the mathematical sense, that is a function that assignes a non-negative number to any (open) interval (a, b). Similarly, g_ν assignes a complex number to any complex number $z \in \mathbb{C}^+$. The first formula in Proposition 4 is an identity which relates $\nu(a, b)$ with an integral over $g_\nu(x + \mathbf{i}y)$. It is a standard formula and we provide a precise reference in the new version of the article.

(32) *Appendix B: the authors could add subsections.*

Two new subsections have been added to the Appendix B:

- B.1 Methods for validating heuristics 1.
- B.2 Spectrum: a computer based approach.

(33) *p.27: The beginning of this page is not clear: why can they simplify assuming that $m_k(z) = \mu(z)$ or $\nu(z)$? They should give more details. Moreover, they should give of precise reference when they are writing about RMT.*

We have added an argument to explain this fact, around Equation (28). We also modified the notations which were misleading (in various parts of the appendix, ν was either a probability measure or a Stieltjes transform!)

(34) *p.27 using "a" the*

This has been fixed.

(35) *Section C.1: the authors should recall that (x_k^*) are assuming to be independent from (B_{kl}), which is a strong assumption, even though it is probably true asymptotically.*

We agree with the reviewer that we should recall the independence assumption. We added the following sentence at the beginning of subsection C.1:

" It should be noted that the assumption of independence between (x_k^*) and (B_{kl}) is a strong one, although it is likely to be true asymptotically, supported by the chaos hypothesis. "

(36) *p.28: I do not see the interest of the equation below the figure.*

We agree that the rationale for this equation is not compelling and that it was only included in an earlier version of the article. As a result, this equation has been removed.

(37) *p.30: the authors should give a reference or more details to get equation (27).*

We concur with the reviewer's assessment that the derivation of equation (27) was not sufficiently clear. To address this, we have added the following sentence with specific reference: " Since $(\check{Z}_k)_{k\in[n]}$ is a family of i.i.d. random variables $\mathcal{N}(0,1)$, using standard extreme value theory of Gaussian random variables (see Leadbetter *et al.* [13]) "

Other remarks

(38) A script, code availability paragraph has been added before the reference section.
(39) A funding paragraph has been added before the reference section.
(40) A conflict of interest disclosure paragraph has been added before the reference section.

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